Thin film of non-Newtonian fluid on an incline

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The slow flow of thin liquid films on solid surfaces is an important phenomenon in nature and in industrial processes, and an intensive effort has been made to investigate it. It is well known that the contact line of currents on an inclined surface may become unstable and then a pattern of ''fingers'' develops that affects the quality of the coatings. This instability has been intensively studied due to its relevance for the technology of various industrial processes. So far the theoretical and numerical research has been focused on Newtonian fluids, notwithstanding that often in the real situations as well as in the experiments, the rheology of the involved liquid is non-Newtonian. Using the lubrication approximation, we derive the governing equations for a current of a power law non-Newtonian fluid on an inclined plane under the action of gravity and the viscous stresses. We show that surface tension effects can be included in the theory by a slight modification of the governing equations, that can then be used as a starting point to investigate the influence of rheology on the fingering instability and other phenomena of interest. We consider the one-dimensional case and we present three families of traveling wave solutions: two running downwards and the other upwards.

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I. INTRODUCTION

The slow flow of a thin film of a liquid is an ubiquitous phenomenon; it occurs in nature as in lava flows, the linings of mammalian lungs, tear films in the eye, and in artificial instances such as microchip fabrication, tertiary oil recovery as well as in many coating processes. Thus, an intensive effort has been spent to achieve a good insight about these types of flows.

The theory of these currents is usually developed within the frame of the lubrication aproximation. Flows on a horizontal plane have been studied theoretically and in the laboratory by several authors (see, for example, Refs. [1–4]). The equations for the same problem but on a general topography have been derived by Buckmaster [5]. Since the experimental research of Huppert [6] and Silvi and Dussan [7] on currents with a contact line on an inclined plane, it is well known that the contact line may become unstable and then a pattern of ''fingers'' develops. This instability has been intensively studied theoretically and numerically (see, Ref. [8] and references therein) due to its relevance for the technology of various industrial processes. All the previously mentioned theoretical and numerical works are based on the assumption that the liquid is Newtonian, notwithstanding the fact that the liquids involved in the real situations and in the experiments are often non-Newtonian. There are few papers where the non-Newtonian behavior is considered. In the recent paper of de Bruyn *et al.* [9], the conditions for the fingering instability of the contact line are investigated for a yield-stress fluid, but the equations that describe the evolution of the free surface are not derived. In Ref. [10] the authors obtained the governing equations of slow gravity flows on a horizontal plane for a fluid with a power law rheology, and in Ref. [11] the flow of a Bingham fluid on an incline is studied. Both works are based on the lubrication approximation. We must also mention in this context the investigation of the stability of the viscoelastic coating flows by Fraysse and Homsy [12] and Spaid and Homsy [13]. In related problems concerning flows over nonhorizontal surfaces, the power law rheology as well as the Bingham model have been assumed in the investigation of roll waves on a shallow layer of fluid mud within the hydraulic approximation [14,15]. The same approximation in conjunction with the viscoplastic Herschel-Bulkley model has been employed to study mud flows down a slope [16]. See also the work of Coussot on roll waves of non-Newtonian fluids [17].

In this paper, we investigate theoretically the slow flow of a power law non-Newtonian liquid on an incline. In Sec. II we derive within the lubrication approximation the governing equations for the evolution of the free surface and the velocity of the fluid when it is under the effect of gravity and viscous stresses. Since the role of surface tension appears to be crucial in the fingering process, and the influence of non-Newtonian behavior on this phenomenon has not been investigated, at the end of this section we show how to modify the governing equations to take into account the surface tension. In Sec. III we consider the simple one-dimensional case, where the flow is downslope or upslope. Besides the two very simple solutions, we find three families of traveling wave solutions that can be obtained in closed form. In addition, we discuss the flow in the limits of a very gentle and a very steep slope. Section IV contains the final discussion.

II. BASIC EQUATIONS

We consider a fluid moving on a nonhorizontal plane, whose angle of inclination is α, as shown in Fig. 1. The

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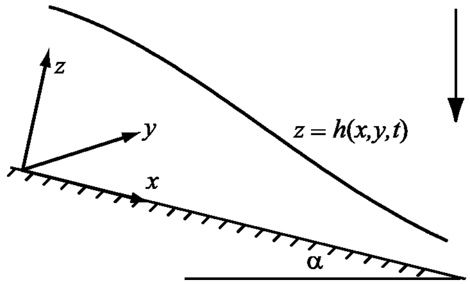

FIG. 1. Geometry of the problem.

coordinate z is perpendicular to the plane, and the x-y coordinates lie in the plane, y is horizontal, and x increases downward. We denote the components of the velocity of the fluid in the directions x, y, and z as u, v, and w, respectively.

Let $z=h\equiv h(x,y,t)$ be the free surface of the current, whose characteristic thickness and characteristic length along the plane we denote with H and L, respectively. The main assumption in the lubrication approximation is

$$H \ll L. \tag{1}$$

From this, it is clear that the component of the velocity parallel to the plane is much larger than the normal component, so that

$$\sqrt{u^2+v^2} \gg |w|. \tag{2}$$

We shall consider sufficiently slow flows so that the inertial effects can be neglected. It will be shown later that this requires that the appropriate Reynolds number must be of the order of unity or less. To derive the governing equations of these flows we shall disregard momentarily the surface tension, so that the liquid is flowing under the action of gravity and the viscous stresses. At the end of this section, we show how these equations must be modified to include capillarity effects.

The equations of momentum balance and the equation of continuity can be written as

$$-\partial_x p + \partial_x \tau_{xx} + \partial_y \tau_{xy} + \partial_z \tau_{xz} + \rho g \sin\alpha = 0, \tag{3}$$

$$-\partial_y p + \partial_x \tau_{yx} + \partial_y \tau_{yy} + \partial_z \tau_{yz} = 0, \tag{4}$$

$$-\partial_z p + \partial_x \tau_{zx} + \partial_y \tau_{zy} + \partial_z \tau_{zz} - \rho g \cos\alpha = 0, \tag{5}$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \tag{6}$$

where p is the pressure, ρ is the density, g is the acceleration of gravity, and τ_{ij} are the components of the deviatoric stress tensor.

We assume a power law constitutive equation of the type [18]

$$\tau_{ij} = 2AE^{(1-\lambda)/\lambda}\dot{\varepsilon}_{ij},$$

$$E = (\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij})^{1/2},$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j), \tag{7}$$

where $\dot{\varepsilon}_{ij}$ are the components of the strain rate tensor, E is its second invariant, and A and λ are positive constants. The power law rheology is the simplest way to model many non-Newtonian fluids of practical interest, and with appropriate choices of A and λ according to the strain rates of the problem at hand, it describes reasonably well their behavior. The Newtonian rheology is retrieved setting $\lambda=1$, in which case A is the viscosity.

We call $U_{||}$ and W the characteristic velocities parallel and perpendicular to the plane, respectively (U and V are the x and y components of $U_{||}$, respectively). From Eq. (6) we find that $U_{||} \gg W = HU_{||}/L$. Using this result, rheological laws (7), and Eq. (1), it is possible to estimate the order of magnitude of each term in Eqs. (3), (4), and (5). Then, retaining only the relevant terms, these equations can be approximated to

$$-\partial_x p + \partial_z \tau_{xz} + \rho g \sin\alpha = 0, \tag{8}$$

$$-\partial_y p + \partial_z \tau_{yz} = 0, \tag{9}$$

$$-\partial_z p - \rho g \cos\alpha = 0. \tag{10}$$

The last equation states that the pressure is hydrostatic as usual in the lubrication approximation. Integrating this equation and using the boundary condition $p=0$ at $z=h$, we find

$$p = \rho g \cos\alpha (h-z). \tag{11}$$

The relevant components of stress tensor are

$$\tau_{xz} = A\{\tfrac{1}{2}[(\partial_z u)^2 + (\partial_z v)^2]\}^{(1-\lambda)/2\lambda}\partial_z u, \tag{12}$$

$$\tau_{yz} = A\{\tfrac{1}{2}[(\partial_z u)^2 + (\partial_z v)^2]\}^{(1-\lambda)/2\lambda}\partial_z y. \tag{13}$$

Replacing Eqs. (11), (12), and (13) in Eqs. (8) and (9), integrating from $z=0$ to $z=h$, and imposing the no stress condition at the free surface ($u_z=v_z=0$ at $z=h$), we obtain

$$[u_z^2+v_z^2]^{(1-\lambda)/2\lambda}u_z = K(\tan\alpha - h_x)(h-z), \tag{14}$$

$$[u_z^2+v_z^2]^{(1-\lambda)/2\lambda}v_z = -Kh_y(h-z), \tag{15}$$

where $K \equiv A^{-1}2^{(1-\lambda)/2\lambda}\rho g \cos\alpha > 0$ and the suffixes of h, u, v denote derivatives. After some algebraic manipulation of these equations it is possible to obtain

$$u_z = (\tan\alpha - h_x)[(\tan\alpha - h_x)^2 + h_y^2]^{(\lambda-1)/2}K^\lambda (h-z)^\lambda. \tag{16}$$

This expression can be integrated in z, and using the no slip condition $u=0$ at $z=0$, we arrive at

$$u=(\tan\alpha-h_x)[(\tan\alpha-h_x)^2+h_y^2]^{(\lambda-1)/2}K^\lambda\frac{h^{\lambda+1}}{\lambda+1}\left[1-\left(1-\frac{z}{h}\right)^{\lambda+1}\right]. \qquad (17)$$

We define the average u as $\overline{u}=h^{-1}\int_0^h u\,dz$. Then

$$\overline{u}=k(\tan\alpha-h_x)[(\tan\alpha-h_x)^2+h_y^2]^{(\lambda-1)/2}h^{\lambda+1}, \qquad (18)$$

where $k\equiv K^\lambda/(\lambda+2)>0$. By means of a similar procedure, we find the expression for $\overline{v}=h^{-1}\int_0^h v\,dz$ as

$$\overline{v}=k(-h_y)[(\tan\alpha-h_x)^2+h_y^2]^{(\lambda-1)/2}h^{\lambda+1}. \qquad (19)$$

Notice that the sign of $\overline{v}$ is opposite to the sign of the slope of the free surface in the y direction. In a similar way, the sign of $\overline{u}$ is opposite to the sign of the slope of the free surface in the x direction, but we must remark that this slope is measured with respect to the horizontal, not with respect to the plane.

Finally, the mass conservation equation can be written in the form

$$h_t+(\overline{u}h)_x+(\overline{v}h)_y=0. \qquad (20)$$

If we insert Eqs. (18) and (19) in the last equation we get

$$\frac{1}{k}h_t+\{(\tan\alpha-h_x)[(\tan\alpha-h_x)^2+(h_y)^2]^{(\lambda-1)/2}h^{\lambda+2}\}_x+\{(-h_y)[(\tan,\alpha-h_x)^2+(h_y)^2]^{(\lambda-1)/2}h^{\lambda+2}\}_y=0. \qquad (21)$$

This equation [or, equivalently, Eqs. (18), (19), and (20)] describes the evolution of the free surface of a power law liquid flowing on an inclined plane in the lubrication approximation.

Surface tension effects can be easily included in the present theory. To this purpose it is sufficient to make in Eq. (21) [or in Eqs. (18), (19), and (20)] the replacements

$$h_x\rightarrow h_x-\frac{\gamma}{\rho g\cos\alpha}(h_{xx}+h_{yy})_x, \qquad (22)$$

$$h_y\rightarrow h_y-\frac{\gamma}{\rho g\cos\alpha}(h_{xx}+h_{yy})_y, \qquad (23)$$

where γ is the surface tension. Using these substitutions in Eq. (21) it can be verified with a little algebra that in the Newtonian case ($\lambda=1$) one obtains the equation studied by Brenner [19].

III. CURRENTS WITH PLANAR SYMMETRY

The solution of the problem described by Eq. (21), including the capillarity by means of substitutions (22) and (23), is clearly very complicated and can only be found numerically except for some very special cases. Here we shall study some special solutions for flows of planar symmetry that can be obtained analytically in closed form. We shall ignore surface tension, and assume that the flow depends only on the x coordinate. Then $h_y=0$ (so that $\overline{v}=0$), and Eq. (21) reduces to

$$h_t+k\sigma\{[\sigma(\tan\alpha-h_x)]^\lambda h^{\lambda+2}\}_x=0, \qquad (24)$$

where $\sigma\equiv\mathrm{sgn}(\overline{u})$. This equation is the generalization to a nonhorizontal plane of an equation obtained by Gratton *et al.* [10].

There are two time-independent solutions of Eq. (24). One of them is the trivial solution $h=(x-x_0)\tan\alpha$ if $x\geqslant x_0$, $h=0$ if $x<x_0$, that represents a static fluid with a horizontal free surface. The other is

$$h=h_0=\text{const} \text{ and } \overline{u}=\overline{u_0}=k(\tan\alpha)^\lambda h_0^{\lambda+1}. \qquad (25)$$

This solution represents a layer of fluid with constant thickness flowing downward with velocity $\overline{u}_0$.

The solutions of Eq. (24) may have fronts (interfaces) and we must distinguish two cases. Indicating with x_f the position of the front, a current limited downslope by a front has $h>0$ for $x<x_f$ and $h=0$ for $x>x_f$, as may occur when the source of the flow is at the top of the incline. Some currents may have a front upslope, in which case $h>0$ for $x>x_f$ and $h=0$ for $x<x_f$. This happens, for example, when the source of the fluid is at the foot of the slope. Assuming that $h\rightarrow0$ when $x\rightarrow x_f$, from Eq. (24) we find two possibile behaviors of $\overline{u}$ near the front: (a) $\overline{u}\rightarrow0$, that is uninteresting, or (b) $h_x\rightarrow\pm\infty$ in a way such that $\overline{u}$ tends to a finite constant different to zero. In this case $\overline{u}\propto|h_x|^\lambda h^{\lambda+1}$. Then, in a neighborhood of the front we have

$$h\propto|x-x_f|^{\lambda/(2\lambda+1)}. \qquad (26)$$

Notice that the surface tension is expected to become important near the front, since there the curvature of the free surface is large. The implications of this fact will be discussed in Sec. IV.

A. Traveling waves

To find traveling wave solutions we assume that h depends on the single variable $s\equiv x-ct$, where c is a constant. Then Eq. (24) can be integrated once to obtain

$$\frac{dh}{ds}=\tan\alpha-\sigma\left[\frac{\sigma(c_1+ch)}{kh^{\lambda+2}}\right]^{1/\lambda}, \qquad (27)$$

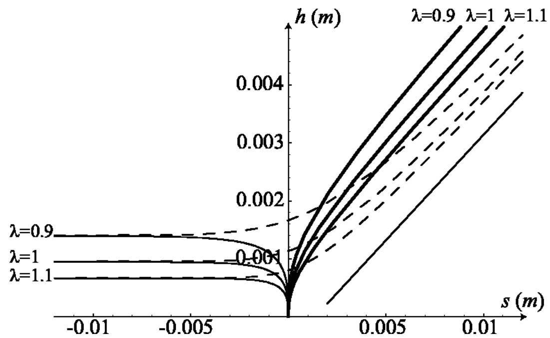

FIG. 2. Profiles of the traveling wave solutions. Thick lines represent the upslope traveling waves; thin lines correspond to the downslope traveling waves with a front and the dashed lines correspond to the waves with no front. The straight oblique line indicates the horizontal. The values of the parameters are $\alpha=20°$, $\rho g/A=9.8\times10^6\ \mathrm{m}^{-1}\ \mathrm{s}^{-1/\lambda}$; they have been chosen so that $\lambda=1$ corresponds to water. All lengths are in meter, and we have assumed $|c|=1$ m/s.

where c_1 is an integration constant. If $c_1=0$ (this implies $\overline{u}=c$), it is possible to integrate Eq. (27) to obtain

$$h-h\,{}_2F_1\left[\mu,1;\mu+1;\sigma\left(\frac{k}{|c|}\right)^{1/\lambda}h^{\frac{1}{\mu}}\tan\alpha\right]=s\tan\alpha+c_2, \tag{28}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function, $\mu=\lambda/(\lambda+1)$, and c_2 is a second integration constant. From this result we obtain three families of solutions, according to the choice of σ and c_2. Some of these solutions have fronts, where h vanishes. Close to the fronts, we find that

$$h=\frac{|c|}{k}\left[-\left(\frac{2\lambda+1}{\lambda}\right)\sigma(s-s_f)\right]^{\lambda/(2\lambda+1)} \tag{29}$$

in agreement with Eq. (26).

We consider first the case $\sigma=+1$ $(c>0)$. We call h_m the solution of

$$c=k(\tan\alpha)^{\lambda}h_m^{\lambda+1}, \tag{30}$$

that is, the same equation that relates $\overline{u_0}$ with h_0 in the steady downslope flow [see Eq. (25)]. The hypergeometric function in Eq. (28) is real for $h\leqslant h_m$ and complex for $h\geqslant h_m$. Then two families of solutions arise:

Downslope traveling waves behind a front. They have $0\leqslant h\leqslant h_m$; in this case we can set $c_2=0$ in Eq. (28), since it can be absorbed into s redefining its origin at the front. For these solutions we have $s<0$, and $h\rightarrow h_m$ for $s\rightarrow-\infty$. Then they represent traveling waves running downslope, that far behind the front $(s\rightarrow-\infty)$ tend to the steady downslope flow [Eq. (25)], and whose profile near the front $(s=0)$ is given by Eq. (29).

Downslope traveling waves with no front. These solutions have $h\geqslant h_m$. The hypergeometric function in Eq. (28) is then complex, and it can be shown that

$$h\,\mathrm{Im}\left[{}_2F_1\left[\mu,1;\mu+1;\left(\frac{h}{h_m}\right)^{1/\mu}\right]\right]=-\pi\mu h_m. \tag{31}$$

Then, setting $\mathrm{Im}(c_2)=\pi\mu h_m$, we obtain a real solution for any s, such that $h\rightarrow h_m$ for $s\rightarrow-\infty$ [so that in this limit it tends to the steady downslope flow Eq. (25)], and $h\rightarrow s\tan\alpha$ for $s\rightarrow+\infty$ (then the free surface tends to the horizontal).

We now consider the case $\sigma=-1$ $(c<0)$ to obtain the third family of solutions:

Upslope traveling waves. We set $c_2=0$ in Eq. (28), since it can be absorbed into s redefining its origin at the front, so that $s>0$, and there is no upper bound of h; for $h\rightarrow\infty$, we have $h=s\tan\alpha$. Thus, in this case Eq. (28) represents a traveling wave solution running upslope with a front like Eq. (29) at $s=0$, and whose profile far from the front tends to the horizontal.

In Fig. 2 we show the three types of solutions for several values of λ.

B. Nearly horizontal and nearly vertical incline

If we assume that $\tan\alpha\ll|h_x|$, Eq. (24) reduces to the governing equation obtained by Gratton *et al.* [10], with g replaced by $g\cos\alpha$. This condition is very restrictive because the lubricating flow approximation requires $|h_x|\sim H/L\ll1$, which means that $\tan\alpha\ll H/L\ll1$.

In the opposite case, when $\tan\alpha\gg|h_x|$, we must set $\sigma=1$ to ensure a real solution. Then Eq. (24) takes the form

$$h_t+c(h)h_x=0\quad\text{with}\quad c(h)=k(\lambda+2)(\tan\alpha)^{\lambda}h^{\lambda+1}. \tag{32}$$

This approximation (32) only describes downslope flows, and is the non-Newtonian generalization of that obtained by Huppert [6].

The general solution of the nonlinear first-order hyperbolic equation (32) can be found by the method of characteristics (see, for example, Ref. [20]). If $f(\xi)$ is an initial profile, then the corresponding solution is given by

$$h=f(x-c(h)t). \tag{33}$$

Therefore, for those values of h such that $f'>0$ (the $'$ denotes the derivative with respect to ξ), the slope h_x will remain positive, tends to zero, thus reinforcing the validity of the assumption $\tan\alpha\gg|h_x|$ that led us to Eq. (32). But for the values of h for which $f'<0$, the wave will break and solution (33) will become multivalued. Breaking will occur first for $t_B=-1/c'(f(\xi_B))$, at the point $x=\xi_B+c(f(\xi_B))t_B$, where ξ_B is determined by the conditions that $c'(f(\xi_B))<0$ and that $|c'(f(\xi_B))|$ is maximum (at the breaking h_x diverges, thus invalidating the assumption $\tan\alpha\gg|h_x|$). All these features can be seen in Fig. 3. It seems reasonable to

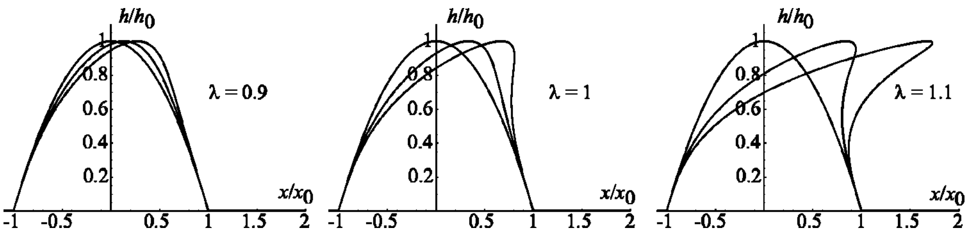

FIG. 3. Solutions of Eq. (32) with an initial profile $f(x)=h_0[1-(x/x_0)^2]$ for $t=0$ s, 0.5 s and 1 s. The parameters are the same as in Fig. 2.

expect that if full equation (24) is used, multivalued solutions will not occur due to the diffusive effect of the term we discarded to arrive to approximation (32).

Finally, it is interesting to show the unique self-similar solution of Eq. (32),

$$h=\left[\frac{x}{(\lambda+2)k(\tan\alpha)^{\lambda}t}\right]^{1/(\lambda+1)}. \tag{34}$$

This solution is the intermediate asymptotic of a wide class of non-self-similar solutions of Eq. (24). For example, the regions of the solutions shown in Fig. 3 with positive slope tend asymptotically to solution (34). Equation (34) is the non-Newtonian counterpart of a solution derived by Huppert [6] for Newtonian liquids.

IV. DISCUSSION

We derived the governing equations of a power law non-Newtonian liquid flowing on an inclined plane, within the lubrication approximation. With the inclusion of surface tension effects, these equations can be used as a starting point to investigate the influence of rheology on the fingering instability and the other phenomena of interest.

We have assumed that inertial effects are negligible, and now we shall see the limitations arising from this assumption. The order of magnitude of the inertial and stress terms are

$$(\vec{v}\cdot\vec{\nabla})\vec{v}\sim\frac{U_{\|}^2}{L}\left(1,1,\frac{H}{L}\right), \tag{35}$$

$$\frac{1}{\rho}\vec{\nabla}\cdot\overline{\overline{\tau}}\sim\frac{AU_{\|}^{1/\lambda}}{\rho H^{(1/\lambda)+1}}\left(1,1,\frac{H}{L}\right). \tag{36}$$

Then the inertia can be neglected if

$$\mathrm{Re}_{\lambda}\equiv\frac{\rho U_{\|}^{2-1/\lambda}H^{1/\lambda}}{A}\sim\mathcal{O}(1), \tag{37}$$

where Re_{λ} is the Reynolds number for a power law fluid [14].

From Eqs. (18) and (19) we notice that due to the non-Newtonian rheology $\overline{u}=\overline{u}(h,h_x,h_y)$ and $\overline{v}=\overline{v}(h,h_x,h_y)$, so that both components of the velocity depend on both components of $\vec{\nabla}h$. In the Newtonian case $(\lambda=1)$ this is not true, since $\overline{u}=\overline{u}(h,h_x)$ and $\overline{v}=\overline{v}(h,h_y)$.

In currents on an incline, gravity plays two roles: it drives the liquid downslope, and it tends to smooth the free surface. This can be seen clearly for the currents with planar geometry, when the y coordinate can be ignored. Then the first effect can be described by the (z-averaged) flow $J_c=k(\tan\alpha)^{\lambda}h^{\lambda+2}$, that is positive, indicating that the liquid runs downslope. It is predominant, where $\tan\alpha\gg|h_x|$. The second effect is described by the diffusive flow $J_d=\sigma_d k(-\sigma_d\partial_x h)^{\lambda}h^{\lambda+2}$, where $\sigma_d=-\mathrm{sgn}(h_x)$. It is important, where $\tan\alpha\ll|h_x|$. However, notice that in the general planar case, the total flow $J=\overline{u}h$ is not a simple combination of J_c and J_d, except in the special case of the Newtonian liquid $(\lambda=1)$. The general expression relating these flows is

$$(\sigma J)^{1/\lambda}=\sigma J_c^{1/\lambda}+\sigma\sigma_d(\sigma_d J_d)^{1/\lambda}. \tag{38}$$

The non-Newtonian rheology is responsible for this nontrivial combination of the gravitational and diffusive effects.

It can be noticed that there is a striking similarity between the traveling wave solutions we found in Sec. III and the threshold profiles of a Bingham fluid on an inclined plane found by Liu and Mei [11] (compare their Fig. 2 with our Fig. 2). In fact, the threshold profiles of Ref. [11] are identical to the profiles of our traveling waves in the case of a Newtonian liquid $(\lambda=1)$.

The solutions we have discussed do not include surface tension effects, which implies that the appropriate Bond number must be large. It is expected that the surface tension will be more relevant where the curvature of the free surface is large, and this occurs close to a front. But precisely there, the lubrication approximation breaks down, so that a correct description of the current near a front cannot be obtained making the replacements of Eqs. (22) and (23) in Eq. (21), but requires a different, more complex approach that is beyond the scope of this paper. We recognize that this problem is also present in the Newtonian case, but, while the shape of a front as given by the lubrication approximation is incorrect, this does not invalidate the remaining parts of the solutions and the spreading relations derived from the theory. In this connection we notice that Goodwin and Homsy [21] have shown that the Newtonian counterpart of Eq. (21) describes adequately the main body of the currents. We do not see any reason why the same should not hold true for non-Newtonian fluids (see Ref. [10]).

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